CALCULUS TOPIC III
COMPLEX NUMBERS

PAUL L. BAILEY

1. Vectors in $\mathbb{R}^2$

1.1. **Arrows.** An *arrow* is a directed line segment in a plane; that is, it is a line segment with one endpoint designated as the *tail* and the other as the *tip*. If the arrow starts at A (A is the tail) and ends at B (B is the tip), we see that such an object is determined by the ordered pair of points (A, B). Denote this arrow by $\widehat{AB}$.

An arrow is determined by three attributes:

- *direction*: this consists of the slope of the line through the two points, together with an orientation along this line;
- *magnitude:* this is the length of the arrow, which is the distance between the tip and the tail;
- *position:* this is described by the placement of the arrow in the plane, and is determined by its tail.

It is possible to define addition of such objects if the tip of one equals the tail of the next; the sum is then defined to be the arrow which starts at the tail of the first and ends at the tip of the second. More precisely,

$$\widehat{AB} + \widehat{BC} = \widehat{AC}.$$

We would like to extend this definition between any two arrows, but in order to do this we need to slide the arrows around via *parallel transport*; that is, we need to disregard the position of the arrow, and only consider its direction and magnitude. We make this precise as follows.

Say that two arrows are *equivalent* if they have the same direction and magnitude, but possibly different positions. Break the set of all arrows in the plane into blocks, where the members of one block consist of all arrows which are equivalent to any other arrow in the block. We call such a block an *equivalence class* of arrows. Any arrow in an equivalence class is called a *representative* of that class.

Date: March 22, 2007.

1.2. **Vectors.** A *vector* is an equivalence class of arrows. The vector represented by an arrow $\widehat{AB}$ is denoted $\overrightarrow{AB}$.

A vector is determined by two attributes:

- direction;
- magnitude.

Thus $\overrightarrow{AB} = \overrightarrow{CD}$ if and only if $\widehat{AB}$ and $\widehat{CD}$ have the same direction and magnitude.

It is now possible to define the addition of any two vectors. Let A, B, C, D be any points on the plane, and consider the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Now $\overrightarrow{CD}$ has a unique representing arrow whose tail is at B, say $\widehat{BE}$. The point E is located geometrically by sliding the arrow $\widehat{CD}$ over so that its tail is at B; or, start at B, and move along the direction of $\widehat{CD}$ for a distance of the magnitude of $\widehat{CD}$, and you will end up at E. Thus $\overrightarrow{CD} = \overrightarrow{BE}$, and it follows our motivation to define

$$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{BE}.$$

We put coordinates on the plane. Let A and B are points on the plane, with coordinates given by $A = (a_1, a_2)$ and $B = (b_1, b_2)$ where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

Now consider two additional points $C = (c_1, c_2)$ and $D = (d_1, d_2)$. A moment's reflection convinces one that $\widehat{AB}$ is equivalent to $\widehat{CD}$ if and only if $d_1 - c_1 = b_1 - a_1$ and $d_2 - c_2 = b_2 - a_2$.

Define $B - A = (b_1 - a_1, b_2 - a_2)$; this is a convenient notation. Let O be the origin, so that $O = (0, 0)$. Let $E = B - A$. Then $\overrightarrow{AB} = \overrightarrow{OE}$. In particular, each vector has a representative which starts at the origin. An arrow whose tail is at the origin is referred to as being in *standard position*. In this way, we can uniquely identify a vector by two real numbers, which are the coordinates of the tip when the tail is at the origin.

Let $\langle x, y\rangle$ denote the vector which is represented by the arrow whose tail is $(0, 0)$ and whose tip is (x, y). The operation of *vector addition* is given by the formula

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle.$$

The *zero vector* is $\langle 0, 0\rangle$; this is the equivalence class of an arrow with no length, that is, of a point. The zero vector serves as an additive identity for vector addition:

$$\langle x, y\rangle + \langle 0, 0\rangle = \langle x + 0, y + 0\rangle = \langle x, y\rangle.$$

Another useful operation on vectors is called *scalar multiplication*. Here we take a representing arrow and stretch it by a factor of t, where $t \in \mathbb{R}$. If t is negative, this reverses the orientation. In coordinates, scalar multiplication is given by

$$t\langle x, y\rangle = \langle tx, ty\rangle.$$

Note that if $t = -1$, scalar multiplication gives the additive inverse of a vector:

$$\langle x, y\rangle + (-1)\langle x, y\rangle = \langle x, y\rangle + \langle -x, -y\rangle = \langle x - x, y - y\rangle = \langle 0, 0\rangle.$$

We now recap and organize what we have discussed.

1.3. **Vector Addition and Scalar Multiplication.** Let $\vec{v}_1 = \langle x_1, y_1 \rangle$ and $\vec{v}_2 = \langle x_2, y_2 \rangle$ be vectors in $\mathbb{R}^2$. Define the *vector addition* by

$$\vec{v}_1 + \vec{v}_2 = \langle x_1 + x_2, y_1 + y_2 \rangle.$$

Geometrically, vector addition produces the diagonal of the parallelogram determined by $\vec{v}_1$ and $\vec{v}_2$.

Let $\vec{v} = \langle x, y \rangle$ and $t \in \mathbb{R}$. Define the *scalar multiplication* of $\vec{v}$ by t to by

$$t\vec{v} = \langle tx, ty \rangle.$$

Geometrically, scalar multiplication stretches the vector $\vec{v}$ by a factor of $|t|$, and if t is negative, it reverses its orientation.

Proposition 1. (Properties of Vector Addition)
Let $\vec{v}$, $\vec{w}$, and $\vec{x}$ be vectors in $\mathbb{R}^2$, and let $a, b \in \mathbb{R}$. Let $\vec{0} = \langle 0, 0 \rangle$. Then

- **(a)** $\vec{v} + \vec{w} = \vec{w} + \vec{v}$; *(Commutativity)*
- **(b)** $(\vec{v} + \vec{w}) + \vec{x} = \vec{v} + (\vec{w} + \vec{x})$; *(Associativity)*
- **(c)** $\vec{v} + \vec{0} = \vec{v}$; *(Existence of an Additive Identity)*
- **(d)** $\vec{v} + (-1)\vec{v} = \vec{0}$; *(Existence of Additive Inverses)*

Proposition 2. (Properties of Scalar Multiplication)
Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$, and let $a, b \in \mathbb{R}$.

- **(a)** $1 \cdot \vec{v} = \vec{v}$; *(Scalar Identity)*
- **(b)** $(ab)\vec{v} = a(b\vec{v})$; *(Scalar Associativity)*
- **(c)** $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$; *(Distributivity of Scalar Mult over Vector Add)*
- **(d)** $(a + b)\vec{v} = a\vec{v} + b\vec{v}$. *(Distributivity of Scalar Mult over Scalar Add)*

Proposition 3. (Additional Properties of Scalar Multiplication)
Let $\vec{v}$ be a vector in $\mathbb{R}^2$, and let $a, b \in \mathbb{R}$. Let $\vec{0} = \langle 0, 0 \rangle$. Then

- **(a)** $0 \cdot \vec{v} = 0$;
- **(b)** $a \cdot \vec{0} = 0$;
- **(c)** $(-a)\vec{v} = -(a\vec{v})$.

We say that two nonzero vectors $\vec{v}$ and $\vec{w}$ are *parallel*, and write $\vec{v} \| \vec{w}$, if arrows representing $\vec{v}$ and $\vec{w}$ lie on parallel line segments. This happens exactly when $\vec{w} = a\vec{v}$ for some nonzero $a \in \mathbb{R}$.

Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$, and suppose we want the vector which proceeds from the tip of $\vec{v}$ to the tip of $\vec{w}$. Call this vector $\vec{x}$. If we follow $\vec{v}$ and then $\vec{x}$, we arrive at the tip of $\vec{w}$; by the geometric interpretation of vector addition, we see that $\vec{v} + \vec{x} = \vec{w}$. Thus $\vec{x} = \vec{w} - \vec{v}$.

Let $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$. Then every vector in $\mathbb{R}^2$ can be written as a sum of scalar multiples of these two vectors; in fact,

$$\langle x, y \rangle = x\vec{i} + y\vec{j}.$$

We call $\vec{i}$ and $\vec{j}$ the *standard basis vectors*.

1.4. **Linear Combinations.** Let $\vec{v}$ and $\vec{w}$ be vectors. A *linear combination* of $\vec{v}$ and $\vec{w}$ is an expression of the form

$$a\vec{v} + b\vec{w},$$

where a and b are real numbers, and first apply scalar multiplication to each vector, and then vector addition.

Given any two nonzero vectors which are not on the same line, it is possible to write every vector as a linear combination of these two. To see this, let $\vec{v}_1 = \langle x_1, y_1$ and $\vec{v}_2 = \langle x_2, y_2$, and consider a third vector $\vec{v} = \langle x, y\rangle$. We wish to find real numbers a_1 and a_2 such that $a_1\vec{v}_1 + a_2\vec{v}_2 = \vec{v}$. Now two vectors in standard position are equal if and only if their x-coordinates and their y-coordinates are equal, so the last vector equation produces a system of two real equations

$$a_1x_1 + a_2x_2 = x;$$
$$a_1y_1 + a_2y_2 = y;$$

this can be solved if the vectors are not on the same line.

1.5. **Norm of a Vector.** The *norm* of a vector is the distance between the tip and the tail of a representing arrow. If the vector is in standard position in $\mathbb{R}^n$, its norm is the distance between the corresponding point and the origin. Thus if $\vec{v} = \langle x, y\rangle$, the norm of $\vec{v}$ is denoted $|\vec{v}|$ and is given by

$$|\vec{v}| = \sqrt{x^2 + y^2}.$$

Synonymous names for this quantity include *modulus*, *magnitude*, *absolute value*, or *length* of the vector.

Example 1. Let $\vec{v} \in \mathbb{R}^2$ be given by $v = \langle 3, 4\rangle$. Find $|\vec{v}|$.

Solution. The length is

$$|\vec{v}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

□

A *unit vector* is a vector whose norm is 1. In some sense, a unit vector represents pure direction (without length); if $\vec{u}$ is a unit vector and a is a scalar, then $a\vec{u}$ is a vector in the direction of $\vec{u}$ with norm a.

Let $\vec{v}$ be any nonzero vector. We obtain a unit vector in the direction of $\vec{v}$ simply by dividing by the length of $\vec{v}$. Thus the *unitization* of $\vec{v}$ is

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v}.$$

Example 2. Let $\vec{v} \in \mathbb{R}^2$ be given by $\vec{v} = \langle 3, 4\rangle$. Find a unit vector in the same direction as v.

Solution. Since $|\vec{v}| = 5$, the unitization of $\vec{v}$ is

$$\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

□

1.6. **Vector Exercises.**

Exercise 1. Draw the directed line segment $\vec{AB}$. Find and draw the equivalent the vector $\vec{v}$ whose tail is at the origin.

(a) $A(3, 1)$, $B(3, 3)$
(b) $A(-3, 5)$, $B(-2, 0)$
(c) $A(0, 2)$, $B(5, 2)$

Exercise 2. Find the vector sum $\vec{v} + \vec{w}$ and illustrate geometrically.

(a) $\vec{v} = \langle 0, 1\rangle$, $\vec{w} = \langle 1, 0\rangle$
(b) $\vec{v} = \langle 2, 4\rangle$, $\vec{w} = \langle 5, 1\rangle$
(c) $\vec{v} = \langle -2, 3\rangle$, $\vec{w} = \langle 3, -2\rangle$

Exercise 3. Find $|\vec{v}|$, $\vec{v} + \vec{w}$, $\vec{v} - \vec{w}$, $2\vec{v}$, and $3\vec{v} - 2\vec{w}$.

(a) $\vec{v} = \langle 1, 2\rangle$, $\vec{w} = \langle 3, 4\rangle$
(b) $\vec{v} = \langle -1, -2\rangle$, $\vec{w} = \langle 2, 1\rangle$
(c) $\vec{v} = \langle 3, 2\rangle$, $\vec{w} = \langle 0, 6\rangle$
(d) $\vec{v} = \vec{i} - \vec{j}$, $\vec{w} = \vec{i} + \vec{j}$

Exercise 4. Find a unit vector which has the same direction as $\vec{v}$.

(a) $\vec{v} = \langle 3, 4\rangle$
(b) $\vec{v} = \langle 5, -5\rangle$
(d) $\vec{v} = \vec{i} + \vec{j}$

Exercise 5. Express $\vec{i}$ and $\vec{j}$ in terms of $\vec{v}$ and $\vec{w}$.

(a) $\vec{v} = \vec{i} + \vec{j}$, $\vec{w} = \vec{i} - \vec{j}$
(b) $\vec{v} = 2\vec{i} + 3\vec{j}$, $\vec{w} = \vec{i} - \vec{j}$

2. Complex Numbers

2.1. **Complex Algebra.** The set of *complex numbers* is

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}, i^2 = -1\}.$$

Let $z_1, z_2 \in \mathbb{C}$. Then $z_1 = x_1 + iy_1$ and $z_2 = x_2 iy_2$ for some $x_1, y_1, x_2, y_2 \in \mathbb{R}$. Define addition and multiplication in $\mathbb{C}$ by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2);$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2).$$

Thus to add or multiply complex numbers, treat i like a variable, add or multiply, replace i^2 with -1, and combine like terms.

One can show that these operations have the following properties:

(F1) $a + b = b + a$ for every $a, b \in \mathbb{C}$;
(F2) $(a + b) + c = a + (b + c)$ for every $a, b, c \in \mathbb{C}$;
(F3) there exists $0 \in \mathbb{C}$ such that $a + 0 = a$ for every $a \in \mathbb{C}$;
(F4) for every $a \in \mathbb{C}$ there exists $b \in \mathbb{C}$ such that $a + b = 0$;
(F5) $ab = ba$ for every $a, b \in \mathbb{C}$;
(F6) $(ab)c = a(bc)$ for every $a, b, c \in \mathbb{C}$;
(F7) there exists $1 \in \mathbb{C}$ such that $a \cdot 1 = a$ for every $a \in \mathbb{C}$;
(F8) for every $a \in \mathbb{C} \setminus \{0\}$ there exists $c \in \mathbb{C}$ such that $ac = 1$;
(F9) $a(b + c) = ab + ac$ for every $a, b, c \in \mathbb{C}$.

Together, these properties state that $\mathbb{C}$ is a *field*. Note that

- $0 = 0 + i0$;
- $1 = 1 + i0$;
- $-(x + iy) = -x + i(-y) = -x - iy$;
- $(x + iy)^{-1} = \dfrac{x - iy}{x^2 + y^2}$.

2.2. **Complex Geometry.** Let $z = x + iy$ be an arbitrary complex number. The *real part* of z is $\Re(z) = x$. The *imaginary part* of z is $\Im(z) = y$. We view $\mathbb{R}$ as the subset of $\mathbb{C}$ consisting of those elements whose imaginary part is zero.

The function $\phi : \mathbb{R}^2 \to \mathbb{C}$ given by $\phi(\langle x, y\rangle) = x + iy$ gives a correspondence between the set of complex numbers and the set of vectors in a plane; the *complex plane* is the set of complex numbers identified with the cartesian plane in this way.

We graph complex number on the xy-plane, using the real part as the first coordinate and the imaginary part as the second coordinate. Thus each complex number is viewed as a vector. Since

$$\begin{aligned}\phi(\langle x_1, y_1\rangle) + \phi(\langle x_2, y_2\rangle) &= (x_1 + iy_1) + (x_2 + iy_2)\\ &= (x_1 + x_2) + i(y_1 + y_2)\\ &= \phi(\langle x_1 + x_2, y_1 + y_2\rangle),\end{aligned}$$

we interpret complex addition geometrically as vector addition. Since

$$a\phi(\langle x, y\rangle) = a(x + iy) = ax + iay = \phi(\langle ax, ay\rangle),$$

we interpret the complex multiplication of a real number times a complex number as scalar multiplication.

Let $z = x + iy$ be an arbitrary complex number. The *conjugate* of z is

$$\overline{z} = x - iy.$$

This is the mirror image of z under reflection across the real axis. Note that

$$z + \overline{z} = (x + iy) + (x - iy) = 2x = 2\Re(z).$$

The *modulus* of z is

$$|z| = \sqrt{x^2 + y^2}.$$

This is the length of z as a vector. Note that

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

The *angle* of z, denoted by $\angle(z)$, is the angle between the vectors $\langle 1, 0\rangle$ and $\langle x, y\rangle$ in the real plane $\mathbb{R}^2$; this is well-defined up to a multiple of 2π.

Let $r = |z|$ and $\theta = \angle(z)$. Then $x = r\cos\theta$ and $y = r\sin\theta$. Define a function

$$\mathrm{cis} : \mathbb{R} \to \mathbb{C} \quad \text{by} \quad \mathrm{cis}(\theta) = \cos\theta + i\sin\theta.$$

Then $z = r\,\mathrm{cis}(\theta)$; this is the *polar representation* of z.

Recall the trigonometric formulae for the cosine and sine of the sum of angles:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \quad \text{and} \quad \sin(A+B) = \cos A\sin B + \sin A\cos B.$$

Let $z_1 = r_1\,\mathrm{cis}(\theta_1)$ and $z_2 = r_2\,\mathrm{cis}(\theta_2)$. Then

$$\begin{aligned}z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\ &= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2))\\ &= r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))\\ &= r_1r_2\,\mathrm{cis}(\theta_1 + \theta_2).\end{aligned}$$

Thus the geometric interpretation of complex multiplication is:

(a) The radius of the product is the product of the radii;
(b) The angle of the product is the sum of the angles.

2.3. **Complex Powers and Roots.** A special case of complex multiplication is exponentiation by a natural number; a simple proof by induction shows that

Theorem 1. (DeMoivre's Theorem)
Let $\theta \in \mathbb{R}$. *Then*

$$(\operatorname{cis}\theta)^n = \operatorname{cis}(n\theta).$$

Let $z = r\operatorname{cis}(\theta)$ and let $n \in \mathbb{N}$. Then $z^n = r^n \operatorname{cis}(n\theta)$.

The *unit circle* in the complex plane is

$$\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Note that if $u_1, u_2 \in \mathbb{U}$, then $u_1 u_2 \in \mathbb{U}$.

Let $\zeta \in \mathbb{C}$ and suppose that $\zeta^n = 1$. We call ζ an n^{th} *root of unity.* Note that if $\zeta = \operatorname{cis}(2\pi/n)$, then

$$\zeta^n = \operatorname{cis}^n(2\pi/n) = \operatorname{cis}(2\pi n/n) = \operatorname{cis}(2\pi) = 1,$$

so n^{th} roots of unity always exist. In fact, for $k \in \mathbb{Z}$, $\zeta^k = \operatorname{cis}(2\pi k/n)$ is also an $({}^{\text{th}}n)$ root of unity, since

$$(\zeta^k)^n = (\zeta^n)^k = 1^k = 1.$$

Moreover, $\zeta^i = \zeta^j$ if and only if $i \equiv j \pmod{n}$, in particular, $\zeta^n = \zeta^0 = 1$. Thus there are exactly n distinct complex numbers which are n^{th} roots of unity; they form the set

$$\mathbb{U}_n = \{\zeta^k \mid \zeta = \operatorname{cis}(2\pi/n), k = 0, 1, \dots, n-1\}.$$

If $\alpha \in \mathbb{U}_n$, we call α a *primitive* n^{th} *root of unity* if $\alpha^j \neq 1$ for $j = 1, \dots, n-1$. If α is a primitive n^{th} root of unity, then $\mathbb{U}_n = \{\alpha^k \mid k = 0, \dots, n-1\}$.

If one graphs the n^{th} roots of unity in the complex plane, the points lie on the unit circle and they are the vertices of a regular n-gon, with one vertex always at the point $1 = 1 + i0$.

Let $z = r\operatorname{cis}(\theta)$. Then z has exactly n distinct n^{th} roots; they are

$$\sqrt[n]{z} = \zeta^m \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta}{n}\right), \quad \text{where} \quad \zeta = \operatorname{cis}\left(\frac{2\pi}{n}\right) \text{ and } m \in \{0, \dots, n-1\}.$$

The algebraic importance of the complex numbers, and the original motivation for their study, is exemplified by the next theorem. This was first conjectured in the 1500's, but was not proven until the doctoral dissertation of Carl Friedrich Gauss in 1799 at the age of 22. Incidentally, he was also the first to prove the constructibility of a regular 17-gon, at an even earlier age.

Theorem 2. (The Fundamental Theorem of Algebra)
Every polynomial with complex coefficients has a zero in $\mathbb{C}$.

From this, it follows that every polynomial with complex coefficients factors completely into the product of linear polynomials with complex coefficients.

3. Complex Equations and Functions

We consider the loci in $\mathbb{C}$ of some simple equations, and some mapping properties of complex valued functions.

Example 3. Horizontal and vertical lines:

- **(a)** The locus of $\Re(z) = a$ is a horizontal line.
- **(b)** The locus of $\Im(z) = b$ is vertical line.
- **(c)** The locus of $z = \overline{z}$ is the real line.
- **(d)** The locus of $z + \overline{z} = 2a$ is the horizontal line $\Re(z) = a$.
- **(e)** The locus of $z - \overline{z} = 2b$ is the vertical line $\Im(z) = b$.

Example 4. Circles:

- **(a)** The locus of $z\overline{z} = r^2$ is a circle of radius r centered at 0.
- **(b)** The locus of the equation $|z - z_0| = r$ is a circle of radius r centered at z_0.

Example 5. Points:

- **(a)** The locus of the equation $z^5 = 1$ consists of the vertices of a regular pentagon inscribed in the unit circle.
- **(b)** The locus of $z^4 - 5z^2 - 36$ is the set $\{\pm 3, \pm 2i\}$.

Example 6. Let $f : \mathbb{R} \to \mathbb{C}$ be given by $f(x) = \operatorname{cis} x$. This function takes the real line and wraps it around the unit circle in the complex plane infinitely many times; each closed interval of the form $[2k\pi, 2(k+1)\pi)$ covers the unit circle once.

Example 7. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = 2z$. Then f dilates the complex plane by a factor of 2.

Example 8. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = iz$. Then f rotates the complex plane by 90 degrees.

Example 9. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = (1+i)z$. Note that $|1 + i| = \sqrt{2}$ and $\angle(1+i) = \frac{\pi}{4}$. Then f dilates the complex plane by a factors of $\sqrt{2}$ and rotates it by 45 degrees.

Example 10. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z^2$. This function maps the upper half plane onto the whole plane, and it also maps the lower half plane onto the whole plane. It takes the plane and wraps it around 0; it is two-to-one everywhere except at 0.

3.1. **Complex Analysis.** The *distance* between complex number z_1 and z_2 is $|z_1 - z_2|$. This is standard distance in the complex plane, and allows us to precisely define what it means for a complex function to be continuous or differentiable; the definitions are identical to the analogous definitions in the real case, but the consequences are far reaching. Moreover, the theory of sequences and series carries over to the complex numbers.

We use the power series expansion of various familiar real-valued functions to motivate the definitions of their complex analogs. In each case, one may use the ratio test to see that the radius of convergence is infinite, so the functions are defined on the entire complex plane.

Define the complex exponential function

$$\exp : \mathbb{C} \to \mathbb{C} \quad \text{by} \quad \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Define the complex sine function by

$$\sin : \mathbb{C} \to \mathbb{C} \quad \text{by} \quad \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

Define the complex cosine function by

$$\cos : \mathbb{C} \to \mathbb{C} \quad \text{by} \quad \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

Note that exp, sin, and cos, when restricted to $\mathbb{R} \subset \mathbb{C}$, are defined so as to be consistent with other definitions of these real functions. In particular, we still have $e = \exp(1)$.

Define $\log : D \to \mathbb{C}$ to be an inverse function of exp, where

$$D = \mathbb{C} \setminus \{x \in \mathbb{R} \mid x < 0\};$$

then log is continuous on D. Note that $\log(1) = e$.

Let $w, z \in \mathbb{C}$, with $w \in D$. We define w^z by

$$w^z = \exp(z \log(w)).$$

Thus $\exp(z) = e^z$.

Compute that

$$\exp(iz) = \cos(z) + i \sin(z).$$

In particular, if z is the complex number $i\theta$, where $\theta \in \mathbb{R}$, we have

Theorem 3. (Euler's Theorem) *Let $\theta \in \mathbb{R}$. Then*

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

3.2. **Complex Exercises.** The *rectangular form* of a complex number is $z = a+bi$. The *polar form* of a complex number is $z = r \operatorname{cis} \theta$.

Exercise 6. Let $z = 7 - 2i$ and $w = 5 + 3i$.
Compute the following, expressed in rectangular form.

- **(a)** $z + w$
- **(b)** $3z - 8w$
- **(c)** zw
- **(d)** $\frac{z}{w}$
- **(e)** $\overline{z}$ and $|z|$

Exercise 7. Find the rectangular and polar forms of all sixth roots of unity.

Exercise 8. Find the rectangular and polar forms of all solutions to the equation $z^6 - 8 = 0$.

Exercise 9. Find the rectangular and polar forms of all solutions to the equation $z^4 - a = 0$, where $a = -\sqrt{192} + 8i$.

Department of Mathematics and CSci, Southern Arkansas University
E-mail address: plbailey@saumag.edu